Sub-Planck phase space structures and Heisenberg-limited measurements

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Outline

- ✓ Phase-space representations of quantum states
- ✓ Quantum metrology: standard quantum limit and Heisenberg limit
- ✓ Measuring and using sub-Planck structures in phase space
- ✓ Applications in cavity QED using motional cat states
- ✓ Applications in ion traps using motional compass states

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References:
- Phys. Rev. A 73, 023803 (2006)
- quant-ph/0608082, to appear in New Journal of Physics

Quantum-enhanced measurements

❑ **Weak forces can be measured with ultra-sensitive precision using judiciously chosen quantum states**

Quantum metrology has recently acquired practical relevance

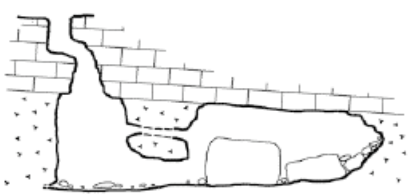
For a review, see V. Giovannetti, and S. Lloyd, L. Maccone, Science **306**, 1330 (2004)

Fundamental science:

Small force measurements (gravity, Casimir forces, etc)

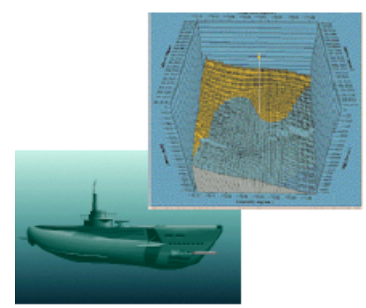
Quantum computation, quantum communication, quantum lithography

Practical uses: gravimetry



Underground Structures

Mineral/Oil Deposits



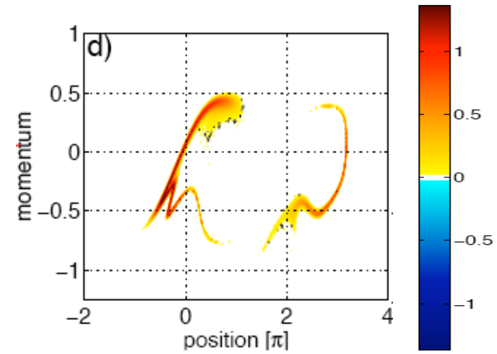
Passive Navigation

Quantum states in phase space

✓ **Classical mechanics**

- **States represented by points/distributions in phase-space:**

$L(x,p) \geq 0$ **classical phase-space distribution (Liouville)**



✓ **Quantum mechanics**

- **States represented by a wave function** $|\Psi\rangle$ **or by a density matrix** $\hat{\rho} = |\Psi\rangle\langle\Psi|$

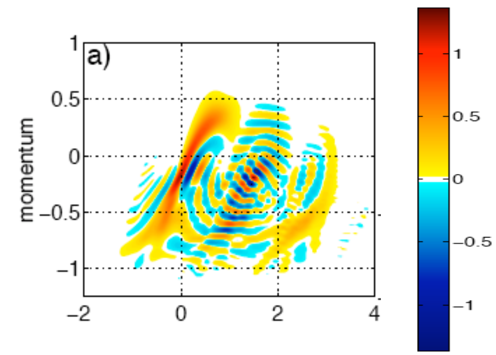
- **Alternatively, they can be represented in phase-space**

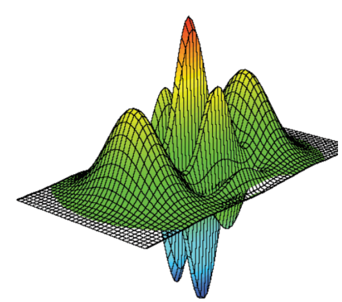
Wigner quantum phase-space distribution:

$$\boxed{W(x,p) = \int \frac{dy}{2\pi\hbar}\, e^{iyp/\hbar}\, \langle x - \frac{y}{2}|\hat{\rho}|x + \frac{y}{2}\rangle}$$

Quasi-probability: <u>can be negative!</u>

Signature of quantum effects (interferences)



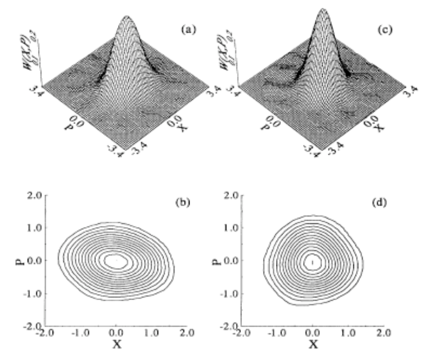


Measurements of the Wigner function

- **Mode of the electromagnetic field**

  Smithey *et al*, PRL **70**, 1244 (1993)
  Breitenbach *et al*, Nature **387**, 471 (1997)

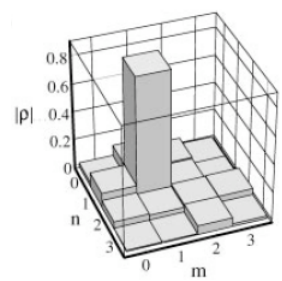
Squeezed

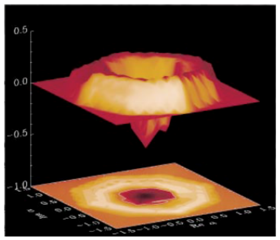


Vacuum

- **Motional quantum state of a trapped ion**

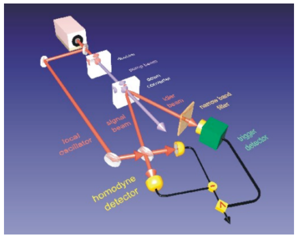
  Wineland group - PRL **77**, 4281 (1996)

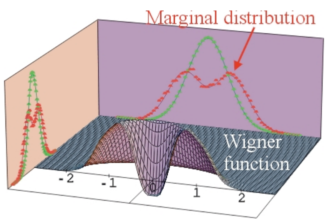




- **One photon Wigner function**

  Lvovsky *et al*, PRL **87**, 050402 (2001)





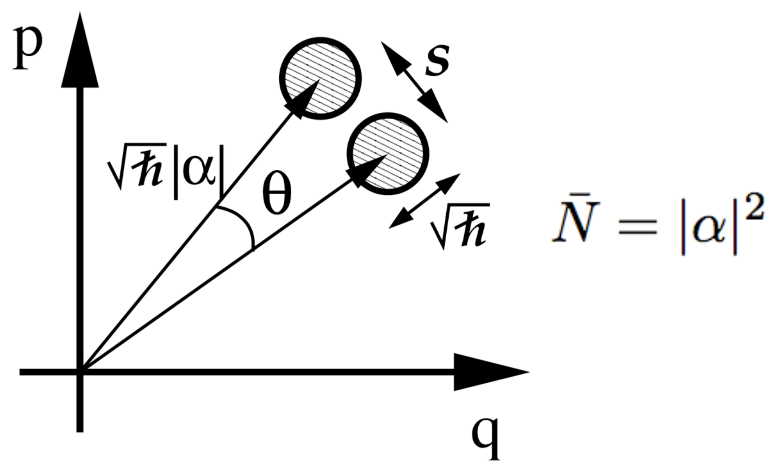
Measuring small displacements and rotations - standard quantum limit

- System is prepared in a known input state $|\Psi\rangle$ which experiences a small displacement, transforming X into $X + s$

- Goal: to infer s with minimum error from measurements performed on the displaced state $|\Psi'\rangle = e^{-is\hat{P}}\,|\Psi\rangle$

- Using probes prepared in *quasi-classical states*, such as coherent states of light, the precision is at the standard quantum limit (SQL)

  - Displacements: $\Delta x \simeq \sqrt{\hbar}$
  - Rotations: $\Delta\theta \simeq \dfrac{\sqrt{\hbar}}{\sqrt{\hbar\bar{N}}} = \dfrac{1}{\sqrt{\bar{N}}}$



This is the naïve guess from Heisenberg uncertainty principle $\Delta x\, \Delta p \geq \hbar/2$

Heisenberg limit

- **Using probes prepared in *quantum correlated states*, such as superposition and entangled states, the precision can be higher that SQL, and reach the ultimate limit allowed by quantum mechanics** $\Rightarrow$ **Heisenberg limit (HL)**

  - **Displacements:** $\Delta x \simeq \frac{\sqrt{\hbar}}{\sqrt{\bar{N}}}$
  - **Rotations:** $\Delta\theta \simeq \frac{1}{\bar{N}}$

$\frac{1}{\sqrt{\bar{N}}}$ **improvement over SQL**

Example: free particle / harmonic oscillator

The mean value of the uncertainty Δx is limited by the mean value of the energy $\langle\Psi|H|\Psi\rangle$

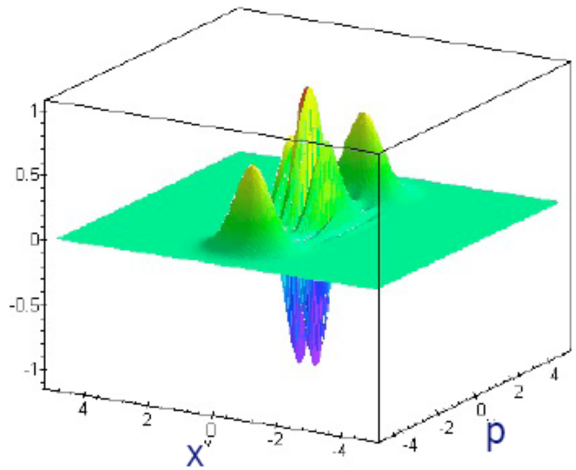
$$\bar{H} = \frac{1}{2}(\overline{P^2} + \overline{X^2}) \geq \frac{\hbar^2}{8(\Delta X)^2} \quad \Rightarrow \quad \Delta X \geq \frac{\sqrt{\hbar}}{\sqrt{8\bar{N}}}$$

Heisenberg limit for displacements

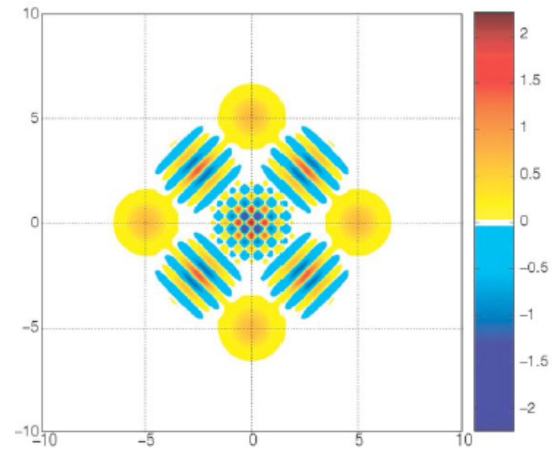
but $\bar{H} = \hbar\bar{N}$

Heisenberg limit and Sub-Planck phase-space structures

The HL is related to sub-Planck phase-space structures (interference)



$$\bar{N} = |\alpha|^2$$



W.H. Zurek, Nature **412**, 712 (2001)

$$|\mathrm{cat}\rangle = (|\alpha\rangle + |-\alpha\rangle)/\sqrt{2}$$

$$|\mathrm{compass}\rangle = (|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle)/2$$

Period of oscillations

$$\Delta x \simeq \frac{\sqrt{\hbar}}{|\alpha|} = \frac{\sqrt{\hbar}}{\sqrt{\bar{N}}}$$

Sub-Planck area of structures

$$\hbar \times \frac{\hbar}{A} \simeq \frac{\hbar}{|\alpha|^2} = \frac{\hbar}{\bar{N}}$$

Sub-shot noise measurements

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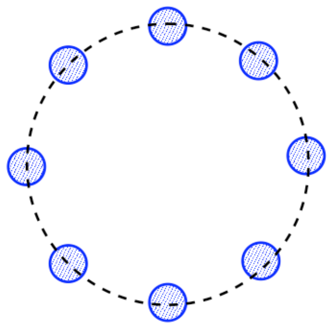
- **Sub-SQL precision has been obtained experimentally using internal degrees of freedom of photons and ions:**
  - ✓ **polarization entanglement with a few photons**
    - Zeilinger group, Nature **429**, 158 (2004)
    - Steinberg group, Nature **429**, 161 (2004)
    - Bouwmeester group, PRL **94**, 090502 (2005)
  - ✓ **Spin entanglement with a few ions**
    - Wineland group, Science **304**, 1476 (2004)
- **Sub-SQL has not been obtained yet using external (motional) degrees of freedom**

Circular coherent states:

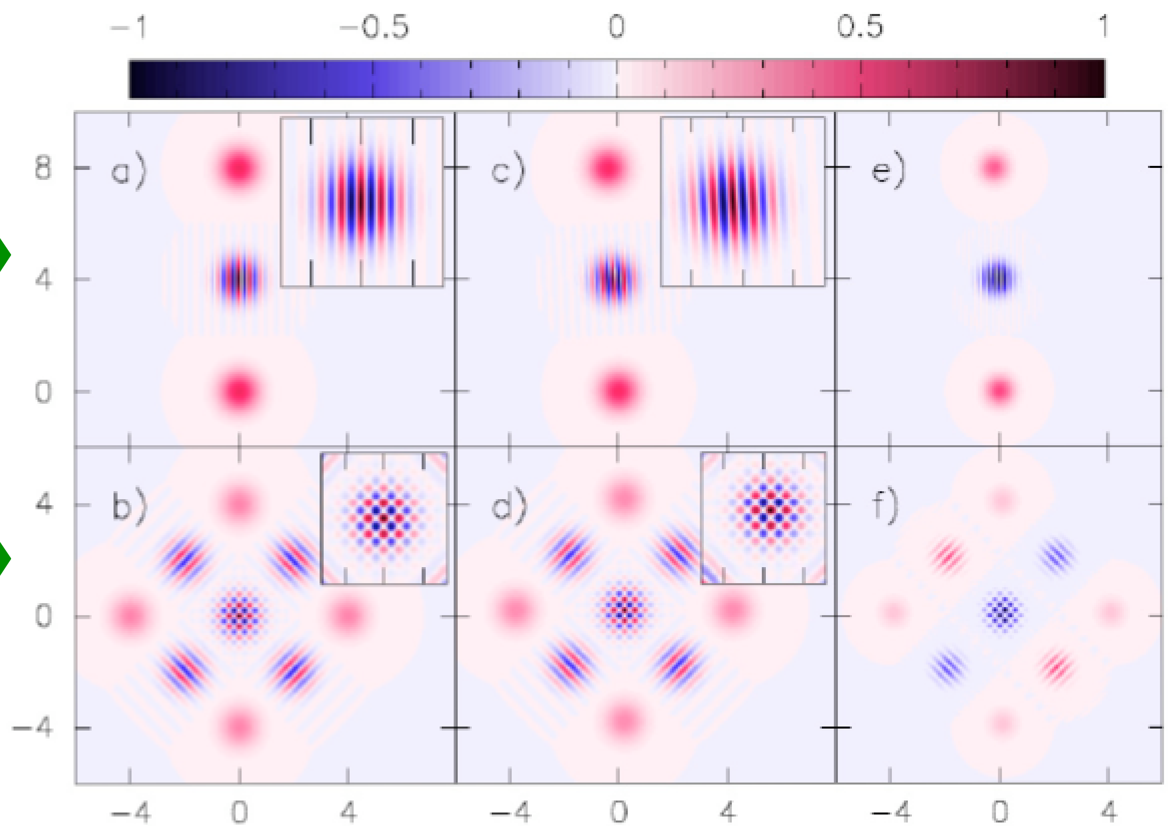
$$\boxed{|\mathrm{cat}_M\rangle = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} e^{i\gamma_k} \, |e^{i\varphi_k}\alpha\rangle}$$



$M = 2 \longrightarrow$ Motional cat state

$M = 4 \longrightarrow$ Motional compass state

Effects of small perturbations



Rotation

Displacement

Minimal rotation

$$\theta_{\text{min}} \simeq \frac{1}{|\alpha|^2}$$

Minimal displacem't

$$s_{\text{min}} \simeq \frac{\sqrt{\hbar}}{|\alpha|}$$

Wigner function of unperturbed state

Wigner function of perturbed state

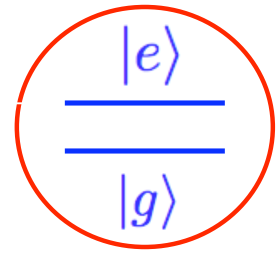
Product of Wigner functions

States can be distinguished

$$\langle \Psi' | \Psi \rangle \approx 0$$

General measurement strategy

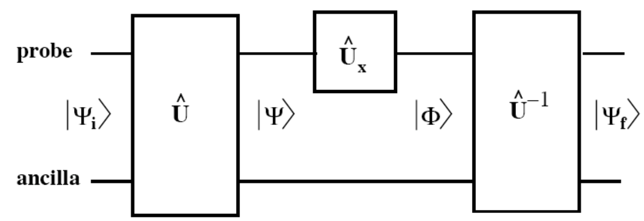
In order to measure the sub-Planck structures, we couple the oscillator to a two-level system (e.g. atomic electronic states, hyperfine levels)



$\hat{U}_x$ (perturbation)

$$|e,\alpha\rangle \xrightarrow{\hat{U}} |\Psi\rangle \xrightarrow{\hat{U}_x} |\Phi\rangle$$

$$\xrightarrow{\hat{U}^\dagger} |\Psi_f\rangle = \sqrt{P_e}|e,\Psi_S^e\rangle + \sqrt{P_g}|g,\Psi_S^g\rangle$$



Note that:

$$|\langle e,\alpha|\Psi_f\rangle|^2 = |\langle\Psi|\Phi\rangle|^2$$

Evolution $\hat{U}$ **must be implemented so that** $|\langle\Psi|\Phi\rangle|^2 = |\langle\mathrm{cat}|\mathrm{cat}(x)\rangle|^2$

$$P_e = 1 - P_g = \frac{|\langle e,\alpha|\Psi_f\rangle|^2}{|\langle\alpha|\Psi_S^e\rangle|^2} = \frac{|\langle\mathrm{cat}|\mathrm{cat}(x)\rangle|^2}{|\langle\alpha|\Psi_S^e\rangle|^2}$$

$\rightarrow$ **Loschmidt echo in quantum systems!**

Cavity QED: two main ingredients

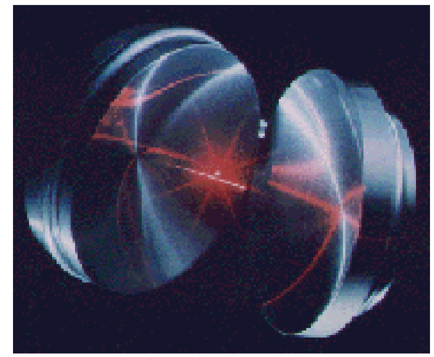
Superconducting mirror cavity

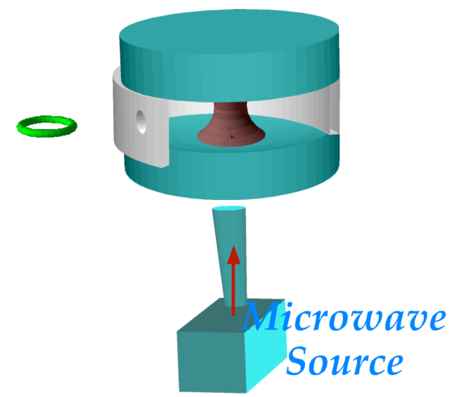
Large field per photon

Long photon life time (0.1 sec)

Easy tunability

Possibility to prepare coherent states with controlled mean photon number





Circular Rydberg atoms

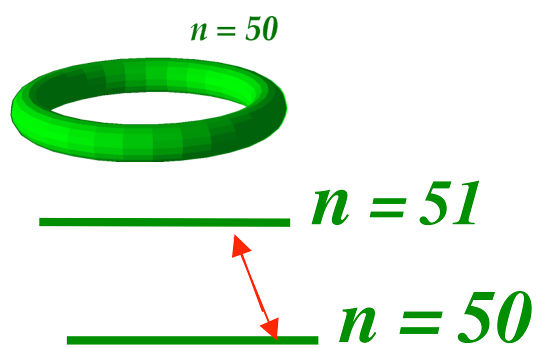
Large circular (classical) orbits

Strong coupling to microwaves

Long radiative lifetimes (30 ms)

Easy state selective detection

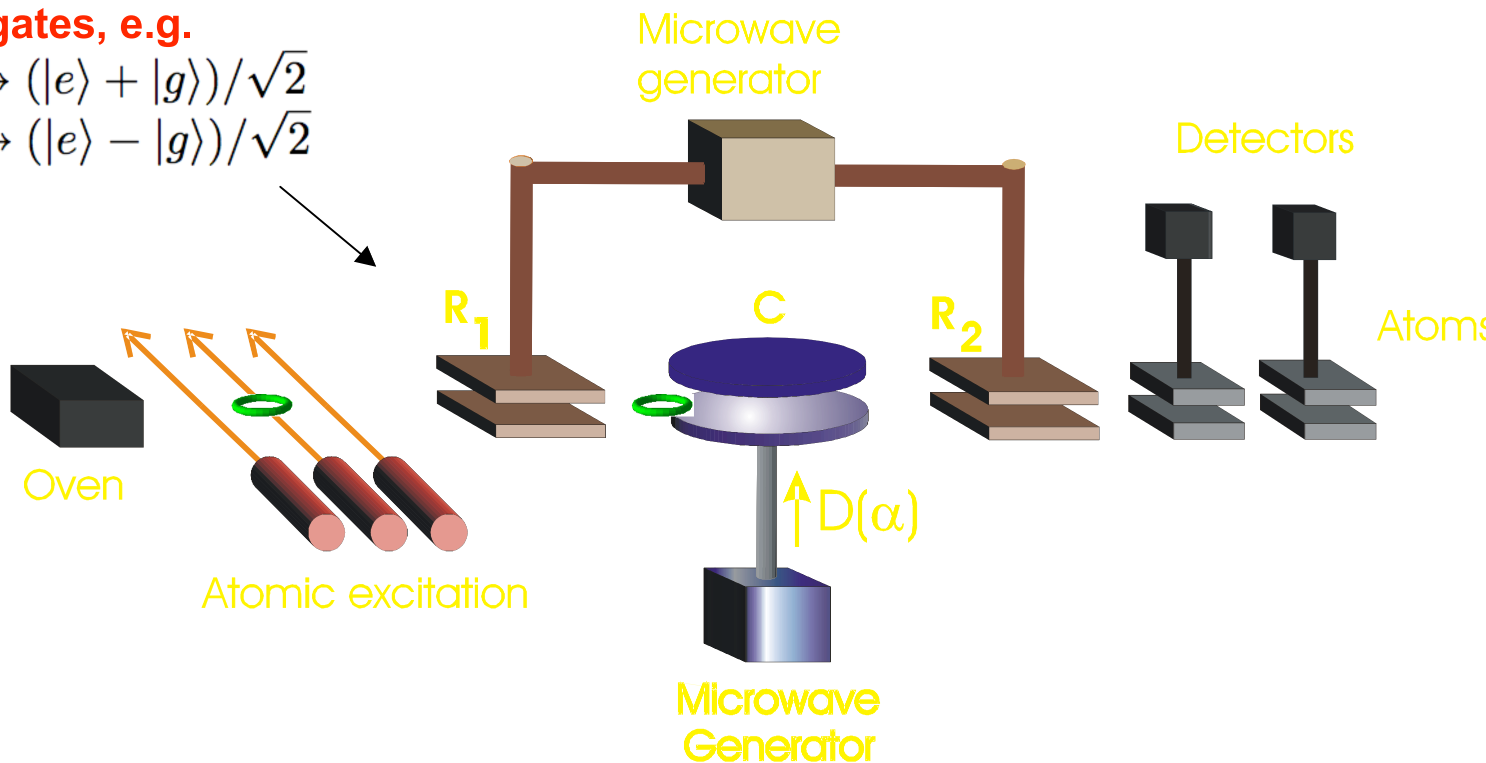
Quasi two-level systems



Cavity QED: EM-field cat states

"1-qubit" gates, e.g.

$$|e\rangle \rightarrow (|e\rangle + |g\rangle)/\sqrt{2}$$
$$|g\rangle \rightarrow (|e\rangle - |g\rangle)/\sqrt{2}$$

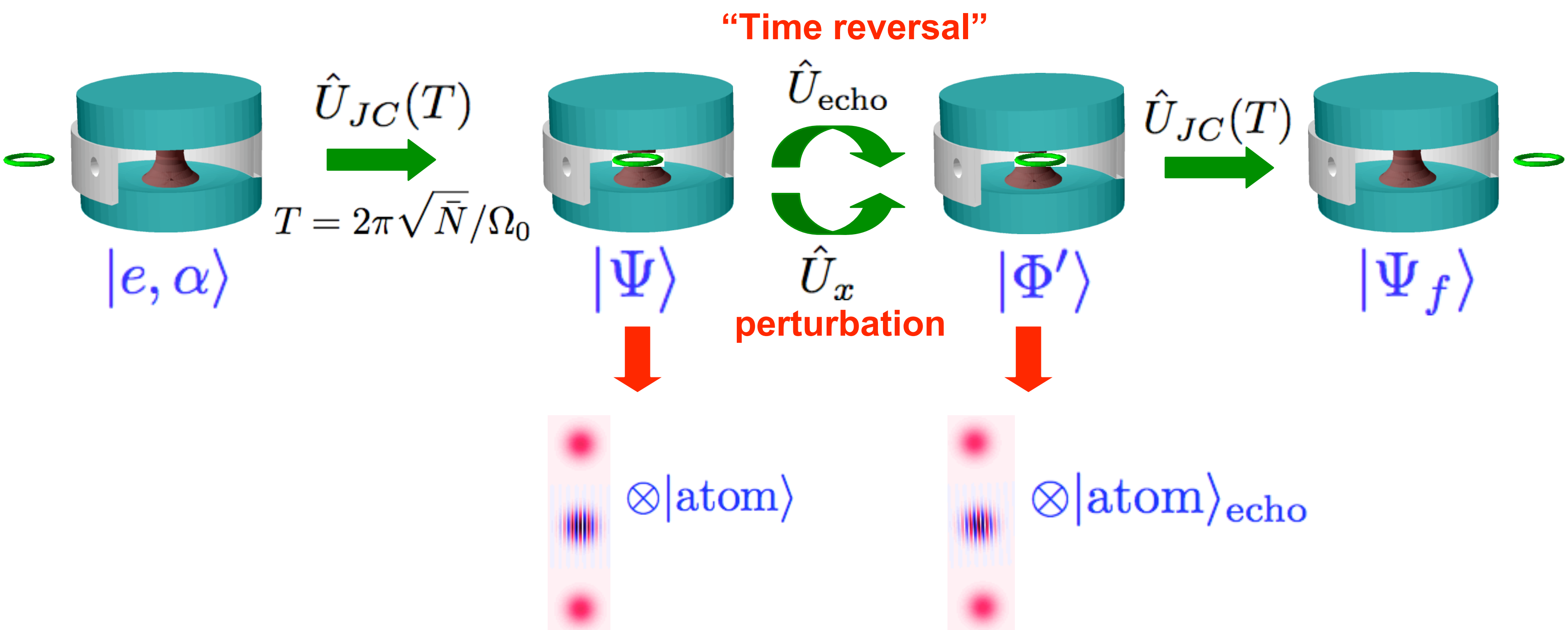


Field - atom interaction (Jaynes-Cummings Hamiltonian)

$$\hat{H}_{JC} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + 1/2\right) + \frac{\hbar\Omega_0}{2}\left(\hat{\sigma}_-\hat{a}^\dagger + \hat{\sigma}_+\hat{a}\right)$$

Resonant interaction

- **Frequency of field = frequency of two-level system** $\omega = \omega_0$



Final state: $|\Psi_f\rangle = \frac{1}{2}\left(1 + e^{i4|\alpha|s}\right)|e,\alpha\rangle + \frac{1}{2}\left(1 - e^{i4|\alpha|s}\right)|g,\alpha\rangle$

Resonant interaction (cont'd)

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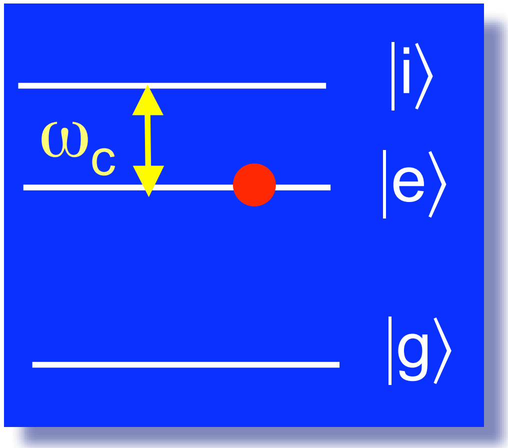
- **How to invert motion?**

  G. Morigi *et al*, PRA **65**, 040102(R) (2002)

  - **Apply percussive 2π pulse to state $|e\rangle$**
  - **Effect on state: $|e\rangle \rightarrow -|e\rangle$**
  - **Effect on JC Hamiltonian: $H_{JC} \rightarrow -H_{JC}$**



- **How to apply perturbation?**

  - **Rotation perturbation: percussive motion of one of the cavity mirrors**
  - **Displacement perturbation: injection of small coherent field into the cavity**

- **Heisenberg-limited measurement**

$$P_e = [1 - \cos(4|\alpha|s)]/2$$

$$\Longrightarrow \quad \Delta s \simeq \frac{1}{\sqrt{\bar{N}}} \quad \Longrightarrow$$

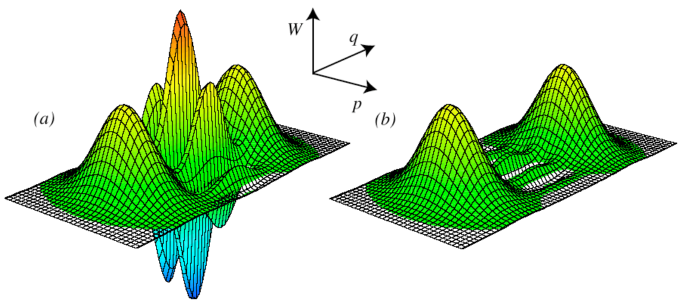
Heisenberg-limited sensitivity to perturbations

Effects of decoherence

- Quantum system interacts with an external environment. This interaction causes loss of coherence (decoherence)

- Quantum superpositions are destroyed

- Main obstacle for coherent quantum dynamics (quantum metrology, etc.)



The main mechanism of decoherence in our proposal is loss of photons from the cavity

$$T \ll T_{\mathrm{dec}} = \tau_{\mathrm{cav}} / \bar{N}$$

$$\Omega_0 = 3 \times 10^5 \mathrm{sec}^{-1} \quad \Rightarrow \tau_{\mathrm{cav}} \gg 1.9 \text{ ms}$$

$$\bar{N} = 20$$

This condition is within reach of present techniques in cavity QED

$\tau_{\mathrm{cav}} \approx 15$ ms (ENS-Paris)

M=2 versus M=4 coherent states

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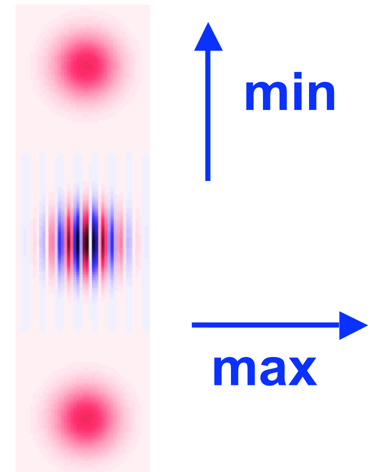
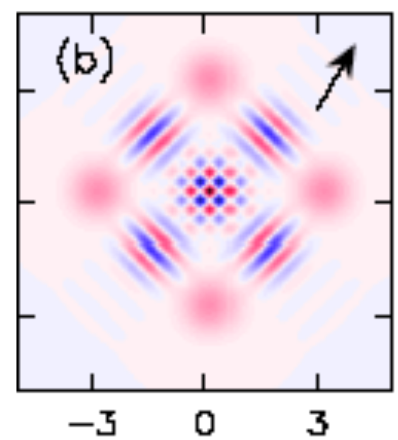
❖ The sensitivity to perturbations of the M=2 states gradually degrades as the direction of the perturbing force moves away from the direction orthogonal to the line joining the two coherent states



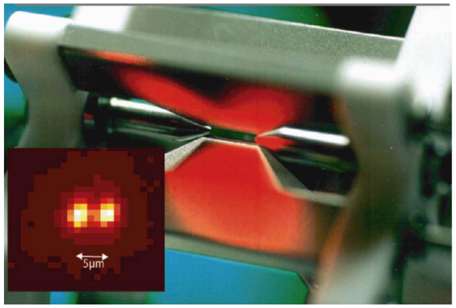
❖ Higher-order (M>2) circular coherent states do not suffer from this limitation

Different proposals in quantum optics for M=4, involving either conditional measurements or dispersive interactions. Their problem is large interaction times.



✓ We propose to do quantum state engineering in ion traps to generate the M=4 state on demand with short interaction times

Ion traps: basic excitation schemes



- Single trapped ion: the center-of-mass motion along each spatial dimension can be described by a quantum harmonic oscillator

- Laser-ion interaction: coupling between internal (electronic) and external (motional) degrees of freedom

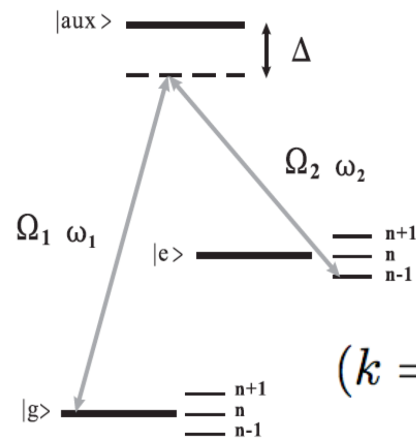
- The laser excitation can be done in several different ways, giving rise to a large number of possible interaction Hamiltonians

Review: D. Liebfried *et al*, Rev. Mod. Phys. **75**, 281 (2003)

- For our purposes, we will consider situations where motional sidebands are spectroscopically well resolved, and the motion along only one principal axis of the trap is effectively excited

First excitation scheme

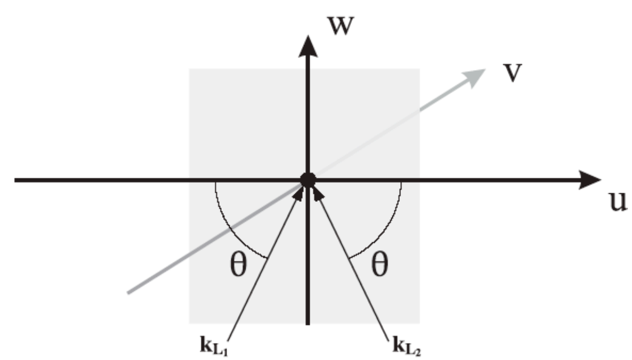
- **Raman excitation of a dipole-forbidden transition on resonance to a given motional sideband**



$$(k = 1)$$

$$\boxed{H_I = \frac{1}{2}\hbar|\Omega_0|e^{i\phi}\ \hat{\sigma}_+\ \hat{f}_k(\hat{n},\eta)\ \hat{a}^k + h.c.}$$

$$\hat{f}_k(\hat{n},\eta) \equiv e^{\eta^2/2}\sum_{l=0}^{\infty}\frac{(i\eta)^{2l+k}}{l!(l+k)!}\ \frac{\hat{n}!}{(\hat{n}-l)!} \qquad \hat{n} = \hat{a}^\dagger\hat{a}$$



Lamb-Dicke parameter: $\eta \equiv (\delta\vec{k}_L \cdot \vec{u})\Delta x_0$

$\eta_{\min} = 0$ for co-propagating lasers

$\eta_{\max}$ for counter-propagating lasers

Effective 1D excitation for $|\vec{k}_{L_1}| \approx |\vec{k}_{L_2}|$ $\quad (\eta_v \approx \eta_w \approx 0)$

First excitation scheme (cont'd)

- **Carrier resonance** $(k=0)$ **and** $\eta \ll 1$

$$H_I = \frac{1}{2}\hbar|\Omega_0|e^{i\phi}\,\hat{\sigma}_+ + h.c.$$

Single qubit rotations

$$\boxed{\hat{U}_\theta(\vec{s}) = e^{-i\theta\vec{s}\cdot\vec{\sigma}}}$$

$$\theta = |\Omega_0|t$$

$$\vec{s} = (\cos\phi, \sin\phi, 0)$$

- **Carrier resonance and larger** η

$$\hat{f}_{k=0} \approx A_0 + A_1\hat{n} \qquad A_0 = 1 - \eta^2/2 \qquad A_1 = -\eta^2$$

Conditional rotations

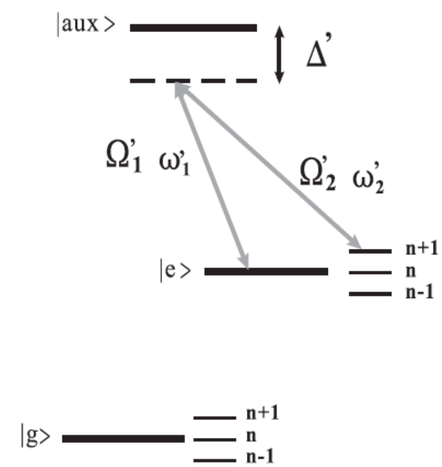
$$\boxed{\hat{R}_c(\bar{\theta}) = e^{i\nu\hat{\sigma}_x}e^{i\bar{\theta}\hat{\sigma}_x\hat{n}}}$$

$$\nu = -|\Omega_0|tA_0/2$$

$$\bar{\theta} = -|\Omega_0|tA_1/2$$

Second excitation scheme

- Raman excitation of one motional sideband via the virtual excitation of a given electronic transition



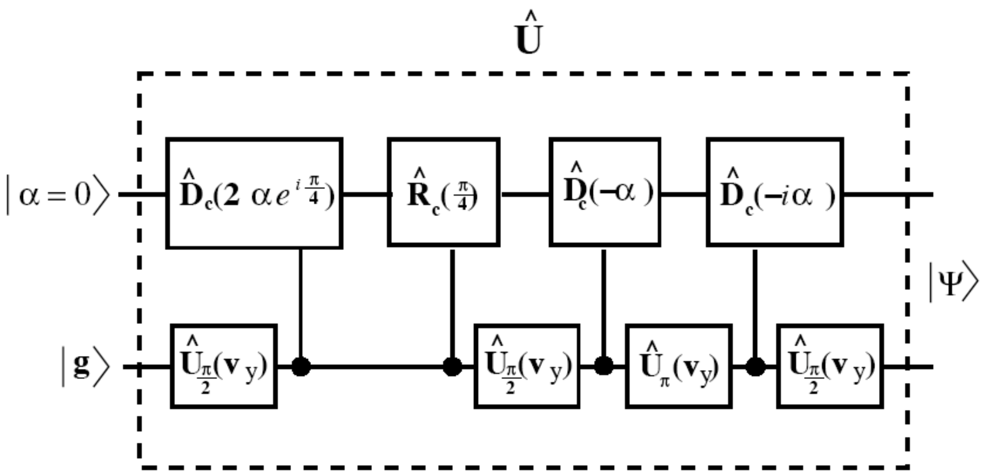
❑ First sideband (k=1) and small Lamb-Dicke parameter

$$H_I = \frac{1}{2}\hbar|\Omega_0|e^{i\phi}\,\hat{a} + h.c.$$

Conditional displacements

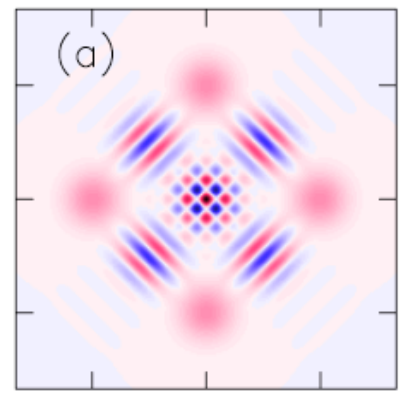
$$\boxed{\hat{D}_c(\alpha) \qquad \alpha = -\frac{1}{2}\eta|\Omega_0|e^{-i\phi}t}$$

Compass state via quantum gates



$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|\mathrm{cat}_4\rangle|e\rangle + |\overline{\mathrm{cat}_4}\rangle|g\rangle]$$

$$|\mathrm{cat}_4\rangle =$$



Experimental parameters [C. Monroe *et al*, Science **272**, 1131 (1996)]

Raman Rabi frequency: $\Omega_0/2\pi = 250\ \mathrm{kHz}$

Lamb-Dicke parameter: $\eta = 0.15$

Time needed to generate the compass state: $t_{\mathrm{cat}_4} \approx 72\mu\mathrm{s}$

Compass state via engineered Kerr nonlinearity

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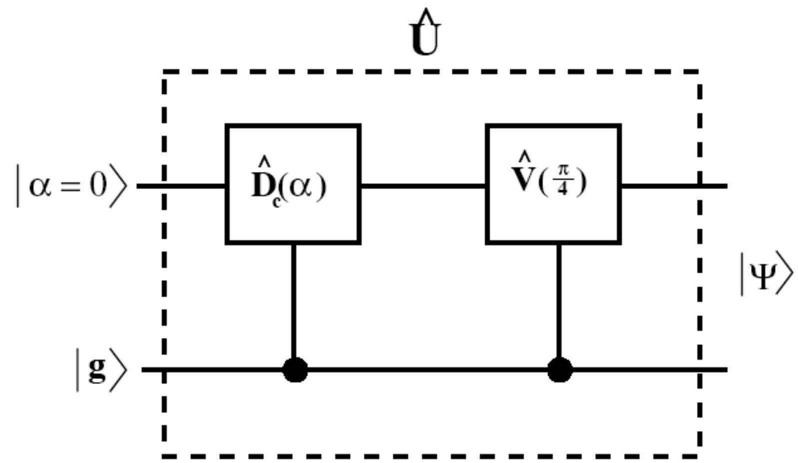
Using two pairs of Raman lasers on carrier resonance, a Kerr-type nonlinearity can be engineered

$$\hat{f}_k^{(e)}(\hat{n},\eta) = A_0 + A_1\hat{n} + A_2\hat{n}^2 + O(\eta_{\max}^8\hat{n}^4)$$

$$\boxed{\hat{V} = e^{-i\phi_0\hat{\sigma}_x}\, e^{-i\phi_1\hat{\sigma}_x\hat{n}}\, e^{-i\phi_2\hat{\sigma}_x\hat{n}^2}}$$



Key identity: $$e^{\pm i(\pi/4)\hat{n}^2}|\alpha\rangle = \frac{1}{2}\left[e^{\pm i(\pi/4)}(|\alpha\rangle - |-\alpha\rangle) + (|i\alpha\rangle + |-i\alpha\rangle)\right]$$

Experimental parameters (new ion traps)

Lamb-Dicke parameters of the two pairs of Raman lasers: $\eta_1 = 0.35$; $\eta_2 = 0.4$

Raman Rabi frequencies: $|\Omega_1| = 5$ MHz ; $|\Omega_2| = 11$ MHz

Time needed to generate the compass state: $t_{\mathrm{cat}_4} \approx 175\mu\mathrm{s}$

Motional decoherence

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The main mechanism of decoherence in this ion trap proposal is heating of the vibrational degree of freedom

- **Typical measured heating times** $\tau_{\text{heating}} \approx 100 \text{ ms}$
- **Estimation of decoherence time of a circular coherent state**

$$\boxed{\tau_{\text{dec}} \propto \frac{\tau_{\text{heating}}}{\bar{n}}}$$

Eg: for a vibrational compass state with $|\alpha| \approx 3$ $\longrightarrow$ $\tau_{\text{dec}} \approx 10 \text{ ms}$

The decoherence time should be much larger than the total interaction time for weak force detection (generation of compass state, application of perturbation, and inversion of the dynamics)

Eg: for a displacement perturbation $\tau_{\text{pert}} \approx 3\mu\text{s}$ [Wineland *et al*, Nature **403**, 269 (2000)]

$$\tau_{\text{int}} \approx 150 - 350\mu\text{s} \ll \tau_{\text{dec}}$$

Measurement of weak perturbations

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- How to apply perturbation?
  - Rotation perturbation: sudden change of the trapping frequency
  - Displacement perturbation: sudden kick to the trap
- Detection

Via shelving, the populations in $|e\rangle$ and $|g\rangle$ are measured, and then the magnitude of the perturbation is inferred with Heisenberg-limited sensitivity

$$P_e = [1 - \cos(4|\alpha|s)]/2 \quad \longrightarrow \quad \Delta s \simeq \frac{1}{\sqrt{\bar{N}}}$$

$\longrightarrow$ Heisenberg-limited sensitivity to perturbations

Summary

✓ Sub-Planck phase-space structures of quantum states are the root for the Heisenberg-limited sensitivity of such states to perturbations

✓ We proposed a general scheme to measure weak perturbations by entangling a quantum oscillator with a two-level system, in such a way that an M circular coherent state of the oscillator is created via the coupled dynamics

✓ We described possible experimental implementations for cat states (M=2), both in cavity QED and ion traps, and for compass states (M=4) in ion traps. They are within reach of present AMO technology

References:
- Phys. Rev. A 73, 023803 (2006)
- quant-ph/0608082, to appear in New Journal of Physics